

RANDOM TIGHT FRAMES

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ABSTRACT. We introduce probabilistic frames to study finite frames whose elements are chosen at random. While finite tight frames generalize orthonormal bases by allowing redundancy, independent, uniformly distributed points on the sphere approximately form a finite unit norm tight frame (FUNTF). In the present paper, we develop probabilistic versions of tight frames and FUNTFs to significantly weaken the requirements on the random choice of points to obtain an approximate finite tight frame. Namely, points can be chosen from any probabilistic tight frame, they do not have to be identically distributed, nor have unit norm. We also observe that classes of random matrices used in compressed sensing are induced by probabilistic tight frames.

1. INTRODUCTION

Frames are basis-like systems that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, or obtain other desirable features unavailable with orthonormal bases. They have proven useful in fields like spherical codes, compressed sensing, signal processing, and wavelet analysis [6, 7, 8, 10, 11, 12, 14, 15, 17, 16, 19, 25]. Tight frames even provide a Parseval type formula similar to orthonormal bases. However, characterizations and constructions of finite tight frames and finite unit norm tight frames (FUNTFs) were needed [6]. A general characterization of all FUNTFs was given by Benedetto and Fickus in [2], where they proved that the FUNTFs are exactly the minimizers of a functional called the frame potential. This was extended to finite tight frames in [30]. Casazza and Fickus have considered the frame potential in the framework of fusion frames [5]. To approximate a FUNTF, Goyal, Vetterli, and Thao considered in [18] n random points on the sphere. In fact, they showed that independent, identically distributed (i.i.d.) points according to the uniform distribution on the sphere asymptotically (as $n \rightarrow \infty$) become a FUNTF.

The present paper is concerned with frames in a probabilistic setting and the generalization of the results of Goyal, Vetterli, and Thao. Our aim is to allow for a more flexible choice of n points while still preserving the asymptotical tight frame property. We first introduce probabilistic frames and adopt many concepts and properties from finite frames to the probabilistic setting. Probabilistic versions of frames, tight frames, Parseval frames, and FUNTFs are developed. After observing that the uniform distribution on the sphere is a probabilistic unit norm tight frame, we extend the results about the random choice of n points on the sphere as follows: in comparison to [18], we are not limited to the uniform distribution and allow for any probabilistic tight frame. Moreover, the points do not have to be identically distributed nor must they lie on a sphere. This means a significant weakening of

2010 *Mathematics Subject Classification.* 42C15.

Key words and phrases. frames, probability, optimal configurations.

the assumptions in [18] and offers much more flexibility. We use this extension to observe that Bernoulli, Gaussian, and sub-Gaussian random matrices, which are used in compressed sensing, fit into this scheme by choosing their rows according to probabilistic tight frames.

To better understand probabilistic tight frames, we minimize the frame potential as introduced by Benedetto and Fickus within a probabilistic setting. In fact, we characterize probabilistic tight frames as minimizers of the probabilistic frame potential, which also generalizes [30]. Relations to spherical t -designs [9, 24] are also discussed.

The outline is as follows: In Section 2, we recall finite frames, the frame potential, and the characterization of its minimizers as derived by Benedetto and Fickus. We also recall the results of Goyal, Vetterli, and Thao about the random choice of n points on the sphere. Section 3 is dedicated to studying probabilistic frames that are introduced in Section 3.1. Well-known properties from finite frames are adopted to the probabilistic setting, and we define and study probabilistic tight frames. We then generalize the results of Goyal, Vetterli, and Thao in Section 3.2. In Section 4, we study the probabilistic frame potential. We show in Section 4.1 that its minimizers are the probabilistic tight frames, and the relations to spherical t -designs are addressed in Section 4.2. Conclusions are given in Section 5.

2. BACKGROUND

A collection of points $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$ is called a *finite frame for \mathbb{R}^d* if there are two constants $0 < A \leq B$ such that

$$(1) \quad A\|x\|^2 \leq \sum_{i=1}^n |\langle x, x_i \rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{R}^d.$$

The constants A and B are called *lower and upper frame bounds*, respectively. In fact, finite frames are the finite spanning sets [7]:

Lemma 2.1. *The sequence $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$ is a finite frame for \mathbb{R}^d if and only if it spans \mathbb{R}^d .*

The frame property can also be expressed by means of operators. Given a collection of n points $\{x_i\}_{i=1}^n$ in \mathbb{R}^d , we call

$$F : \mathbb{R}^d \rightarrow \mathbb{R}^n, \quad x \mapsto (\langle x, x_i \rangle)_{i=1}^n$$

the *analysis operator*. Its adjoint operator

$$F^* : \mathbb{R}^n \rightarrow \mathbb{R}^d, \quad (c_i)_{i=1}^n \mapsto \sum_{i=1}^n c_i x_i$$

is called the *synthesis operator*. If the collection $\{x_i\}_{i=1}^n$ is a finite frame for \mathbb{R}^d , then the *frame operator* $S = F^*F$ is positive, self-adjoint, and invertible [7]. In this case, the following reconstruction formula holds,

$$(2) \quad x = \sum_{j=1}^n \langle S^{-1}x, x_j \rangle x_j = \sum_{j=1}^n \langle x_j, x \rangle S^{-1}x_j, \quad \text{for all } x \in \mathbb{R}^d,$$

and $\{S^{-1}x_i\}_{i=1}^n$, in fact, is a frame too, called the *canonical dual frame*.

Frames, whose lower and upper frame bounds coincide, play a special role, and we call a collection of points $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$ a *finite tight frame for \mathbb{R}^d* if there is a positive constant A such that

$$(3) \quad A\|x\|^2 = \sum_{i=1}^n |\langle x, x_i \rangle|^2, \quad \text{for all } x \in \mathbb{R}^d.$$

The constant A is called the *tight frame bound*. Note that every finite tight frame gives rise to the expansion

$$(4) \quad x = \frac{1}{A} \sum_{i=1}^n \langle x, x_i \rangle x_i, \quad \text{for all } x \in \mathbb{R}^d.$$

In this sense they are a generalization of orthonormal bases. The following lemma summarizes the standard characterizations of tight frames, cf. [7]:

Lemma 2.2. *Let $\{x_i\}_{i=1}^n$ be a collection of vectors in \mathbb{R}^d , and let A be a positive constant. The following points are equivalent:*

- (i) $\{x_i\}_{i=1}^n$ is a finite tight frame for \mathbb{R}^d with frame bound A ,
- (ii) $F^*F = A\mathcal{I}_d$,
- (iii) Equation (4) holds.

If $A = 1$ in (3), then we call $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$ a *finite Parseval frame*. If all elements of a finite tight frame have unit norm, we call them a *finite unit norm tight frame (FUNTF)* for \mathbb{R}^d . Note that a FUNTF that is also Parseval must be an orthonormal basis [7]. In fact, the frame bounds of a FUNTF are given by:

Lemma 2.3 ([18]). *If $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$ is a FUNTF, then the frame bound A equals n/d .*

Every finite frame for \mathbb{R}^d gives rise to a Parseval frame, cf. [7]:

Lemma 2.4. *If $\{x_i\}_{i=1}^n$ is a finite frame for \mathbb{R}^d with frame operator S , then $\{S^{-1/2}x_i\}_{i=1}^n$ is a finite Parseval frame for \mathbb{R}^d .*

The following identity and inequality for Parseval frames have been derived in [1]:

Theorem 2.5 ([1]). *Let $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$ be a finite Parseval frame for \mathbb{R}^d . For every subset $J \subset \{1, \dots, n\} =: \mathcal{N}_n$ and every $x \in \mathbb{R}^d$, we have*

$$\begin{aligned} \sum_{i \in J} |\langle x, x_i \rangle|^2 - \left\| \sum_{i \in J} \langle x, x_i \rangle x_i \right\|^2 &= \sum_{i \in \mathcal{N}_n \setminus J} |\langle x, x_i \rangle|^2 - \left\| \sum_{i \in \mathcal{N}_n \setminus J} \langle x, x_i \rangle x_i \right\|^2, \\ \sum_{i \in J} |\langle x, x_i \rangle|^2 - \left\| \sum_{i \in \mathcal{N}_n \setminus J} \langle x, x_i \rangle x_i \right\|^2 &\geq \frac{3}{4} \|x\|^2. \end{aligned}$$

Given n points $\{x_i\}_{i=1}^n$ on the sphere $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$, the frame potential as introduced by Benedetto and Fickus in [2] is

$$(5) \quad \text{FP}(\{x_i\}_{i=1}^n) = \sum_{i=1}^n \sum_{j=1}^n |\langle x_i, x_j \rangle|^2.$$

For fixed n , they characterized its minimizers:

Theorem 2.6 ([2]). *Let n be fixed and consider the minimization of the frame potential among all collections of n points on the sphere S^{d-1} .*

$n \leq d$: The minimum of the frame potential is n . The minimizers are exactly the orthonormal systems for \mathbb{R}^d with n elements.

$n \geq d$: The minimum of the frame potential is $\frac{n^2}{d}$. The minimizers are exactly the FUNTFs for \mathbb{R}^d with n elements.

The overlap $n = d$ in Theorem 2.6 is not a problem since every FUNTF with $n = d$ elements is an orthonormal basis. Waldron derived an estimate of the frame potential for general points in \mathbb{R}^d , not necessarily on the sphere:

Theorem 2.7 ([30]). *If $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$ are not all zero and $n \geq d$, then*

$$\frac{\sum_{i=1}^n \sum_{j=1}^n |\langle x_i, x_j \rangle|^2}{\left(\sum_{i=1}^n \|x_i\|^2\right)^2} \geq \frac{1}{d},$$

and equality holds if and only if $\{x_i\}_{i=1}^n$ is a finite tight frame for \mathbb{R}^d .

Goyal, Vetterli, and Thao have shown in [18] that independent and uniformly distributed points on the sphere converge towards a FUNTF. To properly formulate the convergence, let $\mathcal{M}(\mathcal{B}, S^{d-1})$ denote the collection of probability measures on S^{d-1} with respect to the induced Borel σ algebra \mathcal{B} . If $Z : S^{d-1} \rightarrow U \subset \mathbb{R}^{p \times q}$ is a random matrix/vector, distributed according to $\mu \in \mathcal{M}(\mathcal{B}, S^{d-1})$, then we simply write $Z \in U$ for notational convenience. The expectation of Z is defined by $E(Z) := \int_{S^{d-1}} Z(x) d\mu(x)$, where the integral is taken component-wise. Note that for a collection of random vectors $\{X_i\}_{i=1}^n \subset S^{d-1}$, the frame operator is a random matrix.

Theorem 2.8 ([18]). *For any n , let $\{X_{k,n}\}_{k=1}^n \subset S^{d-1}$ be a collection of n random vectors, i.i.d. according to the uniform probability distribution on the sphere. If F_n denotes the random matrix associated to the analysis operator of $\{X_{k,n}\}_{k=1}^n$, then the matrix operator $\frac{1}{n} F_n^* F_n$ converges towards $\frac{1}{d} \mathcal{I}_d$ in the mean squared sense, i.e., $E(\|\frac{1}{n} F_n^* F_n - \frac{1}{d} \mathcal{I}_d\|_{\mathcal{F}}^2) \rightarrow 0$, where $\|\cdot\|_{\mathcal{F}}$ denotes the Frobenius norm.*

Note that $\frac{1}{n} F_n^* F_n = \frac{1}{d} \mathcal{I}_d$ would mean that we have a FUNTF, cf. Lemma 2.2. In the present paper, we develop a framework that leads to a significant generalization of Theorem 2.8.

3. PROBABILISTIC FRAMES

3.1. Probabilistic Tight Frames. In this section, we shall introduce a probabilistic analogue of finite frames. Let K be a nonempty subset of \mathbb{R}^d and let $\mathcal{M}(\mathcal{B}, K)$ denote the collection of probability measures on K with respect to the induced Borel σ algebra \mathcal{B} .

Definition 3.1. A probability measure $\mu \in \mathcal{M}(\mathcal{B}, K)$ is called a *probabilistic frame* for \mathbb{R}^d if there are constants $0 < A \leq B$ such that

$$A\|x\|^2 \leq \int_K |\langle x, y \rangle|^2 d\mu(y) \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{R}^d.$$

The constants A and B are called *lower and upper probabilistic frame bounds*, respectively. If only the upper inequality holds, then we call μ a *Bessel measure*. A probabilistic frame μ for \mathbb{R}^d is called a *probabilistic unit norm frame* if $K = S^{d-1}$.

It should be mentioned that Definition 3.1 is not entirely new, but constitutes a shift of perspective:

Remark 3.2. In standard continuous frame theory, the measure μ is fixed and elements in a Hilbert space form the frame that is indexed by a continuous set. Definition 3.6 means a shift of perspective because we identify the index set with the elements in the Hilbert space and hold them fixed (to be K). We now allow the measure μ to vary, which then encodes the frame.

If $\{x_i\}_{i=1}^n$ is a frame for \mathbb{R}^d , then the normalized counting measure $\frac{1}{n}\mu_{x_1, \dots, x_n}$ is a probabilistic frame for \mathbb{R}^d with respect to any subset K that contains $\{x_i\}_{i=1}^n$. Thus, Definition 3.1 extends the concept of finite frames for \mathbb{R}^d .

The support of $\mu \in \mathcal{M}(\mathcal{B}, K)$ is

$$\text{supp}(\mu) = \{x \in K : \mu(U_x) > 0, \text{ for all open subsets } U_x \subset K \text{ that contain } x\},$$

and the following is the probabilistic counterpart of Lemma 2.1:

Proposition 3.3. *Assume that $K \subset \mathbb{R}^d$ is bounded. A probability measure $\mu \in \mathcal{M}(\mathcal{B}, K)$ is a probabilistic frame for \mathbb{R}^d if and only if its support spans \mathbb{R}^d .*

Proof. If the support does not span \mathbb{R}^d , then there exists an element $x \in \text{supp}(\mu)^\perp$ that satisfies $\int_K |\langle x, y \rangle|^2 d\mu(y) = 0$. Therefore, μ cannot be a probabilistic frame.

For the reverse implication, we observe that the Cauchy-Schwartz inequality yields

$$(6) \quad \int_K |\langle x, y \rangle|^2 d\mu(y) \leq \sup_{y \in K} (\|y\|^2) \|x\|^2, \quad \text{for all } x \in \mathbb{R}^d.$$

Since K is bounded, μ is a Bessel measure and the upper probabilistic frame bound B exists. To find a lower probabilistic frame bound, let us define

$$A := \inf_{x \in \mathbb{R}^d} \left(\frac{\int_K |\langle x, y \rangle|^2 d\mu(y)}{\|x\|^2} \right) = \inf_{x \in S^{d-1}} \left(\int_K |\langle x, y \rangle|^2 d\mu(y) \right).$$

Due to the dominated convergence theorem, the mapping $x \mapsto \int_K |\langle x, y \rangle|^2 d\mu(y)$ is continuous and the infimum is in fact a minimum since S^{d-1} is compact. Let x be in S^{d-1} such that

$$A = \int_K |\langle x, y \rangle|^2 d\mu(y).$$

Since $\text{supp}(\mu)$ spans \mathbb{R}^d , x cannot be in the orthogonal complement of $\text{supp}(\mu)$, and thus there is $y_0 \in \text{supp}(\mu)$ such that $|\langle x, y_0 \rangle|^2 > 0$. Therefore, there is $\varepsilon > 0$ and an open subset $U_{y_0} \subset K$ satisfying $y_0 \in U_{y_0}$ and $|\langle x, y \rangle|^2 > \varepsilon$, for all $y \in U_{y_0}$. Since $\mu(U_{y_0}) > 0$, we obtain $A \geq \varepsilon \mu(U_{y_0}) > 0$, which concludes the proof. \square

The *analysis operator*

$$F : \mathbb{R}^d \rightarrow L_2(K, \mu), \quad x \mapsto \langle x, \cdot \rangle_{\mathbb{R}^d}$$

is bounded with norm less than or equal to $\sup_{y \in K} (\|y\|^2)$ if and only if (6) holds. We call the adjoint operator

$$F^* : L_2(K, \mu) \rightarrow \mathbb{R}^d, \quad f \mapsto \int_K f(x) x d\mu(x)$$

the *synthesis operator*, where the integral is vector valued. If $\mu \in \mathcal{M}(\mathcal{B}, K)$ is a probabilistic frame for \mathbb{R}^d with *frame operator* $S = F^*F$, then S is positive, self-adjoint, and invertible. Moreover, for $\tilde{\mu} = \mu \circ S$, we obtain

$$(7) \quad y = \int_{S^{-1}K} Sz \langle z, y \rangle d\tilde{\mu}(z) = \int_{S^{-1}K} z \langle Sz, y \rangle d\tilde{\mu}(z), \quad \text{for all } y \in \mathbb{R}^d,$$

which follows from $S^{-1}S = SS^{-1} = \mathcal{I}_d$. In fact, if $\mu \in \mathcal{M}(\mathcal{B}, K)$ is a probabilistic frame for \mathbb{R}^d , then $\tilde{\mu} \in \mathcal{M}(S^{-1}\mathcal{B}, S^{-1}K)$ is a probabilistic frame for \mathbb{R}^d . Note that if μ is the counting measure corresponding to a FUNTF $\{x_i\}_{i=1}^n$, then $\tilde{\mu}$ is the counting measure associated to the canonical dual frame of $\{x_i\}_{i=1}^n$, and Equation (7) reduces to (2). These observations motivate the following definition:

Definition 3.4. If $\mu \in \mathcal{M}(\mathcal{B}, K)$ is a probabilistic frame with frame operator S , then $\tilde{\mu} = \mu \circ S \in \mathcal{M}(S^{-1}\mathcal{B}, S^{-1}K)$ is called the *probabilistic canonical dual frame* of μ .

Remark 3.5. The frame operator S_1 of a finite frame $\{x_i\}_{i=1}^n$ has a different normalization than the frame operator S_2 of the associated normalized counting measure $\frac{1}{n}\mu_{x_1, \dots, x_n}$. In fact, we have $S_2 = \frac{1}{n}S_1$.

Next, we generalize finite tight frames:

Definition 3.6. A probability measure $\mu \in \mathcal{M}(\mathcal{B}, K)$ is called a *probabilistic tight frame* for \mathbb{R}^d if there is a positive constant $0 < A$ such that

$$(8) \quad A\|x\|^2 = \int_K |\langle x, y \rangle|^2 d\mu(y), \quad \text{for all } x \in \mathbb{R}^d.$$

We call μ a *probabilistic Parseval frame* for \mathbb{R}^d if (8) holds with $A = 1$. The probability measure μ is called a *probabilistic unit norm tight frame* for \mathbb{R}^d if it is a probabilistic tight frame with $K = S^{d-1}$.

The following lemma is the probabilistic version of Lemma 2.2 and can be derived from results in continuous frame theory:

Lemma 3.7. Let $\mu \in \mathcal{M}(\mathcal{B}, K)$ and let A be a positive constant. The following points are equivalent:

- (i) μ is a probabilistic tight frame with frame bound A ,
- (ii) $F^*F = A\mathcal{I}_d$,
- (iii) $x = \frac{1}{A} \int_K \langle x, y \rangle y d\mu(y)$, for all $x \in \mathbb{R}^d$.

Many properties of finite frames can be carried over. For instance, we can follow the lines in [7] to derive a generalization of Lemma 2.4:

Proposition 3.8. If $\mu \in \mathcal{M}(\mathcal{B}, K)$ is a probabilistic frame for \mathbb{R}^d , then $\mu \circ S^{1/2} \in \mathcal{M}(S^{-1/2}\mathcal{B}, S^{-1/2}K)$ is a probabilistic Parseval frame for \mathbb{R}^d .

Only the frame operator and associated operators are used in the proof of Theorem 2.5 in [1]. Therefore, we can follow those lines and obtain the fundamental identity and inequality of probabilistic Parseval frames:

Proposition 3.9. Let $\mu \in \mathcal{M}(\mathcal{B}, K)$ be a probabilistic Parseval frame for \mathbb{R}^d . For every measurable subset $J \subset K$ and every $x \in \mathbb{R}^d$, we have

$$\begin{aligned} \int_J |\langle x, y \rangle|^2 d\mu(y) - \left\| \int_J \langle x, y \rangle y d\mu(y) \right\|^2 &= \int_{K \setminus J} |\langle x, y \rangle|^2 d\mu(y) - \left\| \int_{K \setminus J} \langle x, y \rangle y d\mu(y) \right\|^2, \\ \int_J |\langle x, y \rangle|^2 d\mu(y) - \left\| \int_{K \setminus J} \langle x, y \rangle y d\mu(y) \right\|^2 &\geq \frac{3}{4} \|x\|^2. \end{aligned}$$

If $\{x_i\}_{i=1}^n \subset \mathbb{R}^d$ are pairwise distinct vectors, that form a finite tight frame for \mathbb{R}^d , then the normalized counting measure $\frac{1}{n}\mu_{x_1, \dots, x_n}$ is a probabilistic tight frame for \mathbb{R}^d . Lemma 2.3 can also be carried over:

Lemma 3.10. *If $\mu \in \mathcal{M}(\mathcal{B}, K)$ is a probabilistic tight frame for \mathbb{R}^d , then the frame bound A equals $\frac{1}{d} \int_K \|x\|^2 d\mu(x)$.*

Proof. If e_1, \dots, e_d is the canonical basis for \mathbb{R}^d , then we have $Ad = \sum_{i=1}^d A\|e_i\|^2$. The equality (8) and finally the Parseval equality for orthonormal bases yield

$$Ad = \sum_{i=1}^d \int_K |\langle e_i, x \rangle|^2 d\mu(x) = \int_K \sum_{i=1}^d |\langle e_i, x \rangle|^2 d\mu(x) = \int_K \|x\|^2 d\mu(x). \quad \square$$

For $\mu \in \mathcal{M}(\mathcal{B}, K)$, one easily verifies that the frame operator $S = F^*F$ is given by

$$F^*F : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad F^*F(x) = \int_K \langle x, y \rangle y d\mu(y).$$

If $\{e_i\}^d$ is the canonical basis for \mathbb{R}^d , then the vector valued integral yields

$$\int_K y^{(i)} y d\mu(y) = \sum_{j=1}^d \int_K y^{(i)} y^{(j)} d\mu(y) e_j,$$

where $y = (y^{(1)}, \dots, y^{(d)})^\top \in \mathbb{R}^d$. If we denote the second moments of μ by $m_{i,j}(\mu)$, i.e.,

$$m_{i,j}(\mu) = \int_K x^{(i)} x^{(j)} d\mu(x), \quad \text{for } i, j = 1, \dots, d,$$

then we obtain

$$F^*F e_i = \int_K y^{(i)} y d\mu(y) = \sum_{j=1}^d \int_K y^{(i)} y^{(j)} d\mu(y) e_j = \sum_{j=1}^d m_{i,j}(\mu) e_j.$$

Thus, the frame operator is the matrix of second moments. As a consequence, Lemma 3.7 implies the following characterization of probabilistic tight frames:

Corollary 3.11. *A probability measure $\mu \in \mathcal{M}(\mathcal{B}, K)$ is a probabilistic tight frame for \mathbb{R}^d if and only if its second moments satisfy*

$$(9) \quad m_{i,j}(\mu) = \frac{1}{d} \delta_{i,j} \int_K \|x\|^2 d\mu(x), \quad \text{for all } i, j = 1, \dots, d.$$

Remark 3.12. Bourgain raised in [4] the following question: *Is there a universal constant $c > 0$ such that for any dimension d and any convex body K in \mathbb{R}^d with $\text{vol}_d(K) = 1$, there exists a hyperplane $H \subset \mathbb{R}^d$ for which $\text{vol}_{d-1}(K \cap H) > c$?* The positive answer to this question has become known as the hyperplane conjecture. By applying results in [23], we can rephrase this conjecture by means of probabilistic tight frames: *There is a universal constant C such that for any convex body K , on which the uniform probability measure σ_K forms a probabilistic tight frame, the probabilistic tight frame bound is less than C .* Due to Lemma 3.10, the boundedness condition is equivalent to $\int_K \|x\|^2 d\sigma_K(x) \leq Cd$. The hyperplane conjecture is still open, but there are large classes of convex bodies, for instance, gaussian random polytopes [21], for which an affirmative answer has been established.

Let us further investigate the uniform probability measure:

Proposition 3.13. *The uniform probability measure σ_r on the sphere of radius $r > 0$ is a probabilistic tight frame.*

Proof. We aim to verify the conditions in Corollary 3.11. First, we consider $i \neq j$: we divide the sphere $S_r^{d-1} = \{x \in \mathbb{R}^d : \|x\| = r\}$ into four parts,

$$\begin{aligned} P_1 &= \{x \in S_r^{d-1} : 0 \leq x^{(i)}, x^{(j)} \leq 1\}, \\ P_2 &= \{x \in S_r^{d-1} : 0 \leq x^{(i)}, -x^{(j)} \leq 1\}, \\ P_3 &= \{x \in S_r^{d-1} : 0 \leq -x^{(i)}, x^{(j)} \leq 1\}, \\ P_4 &= \{x \in S_r^{d-1} : 0 \leq -x^{(i)}, -x^{(j)} \leq 1\}. \end{aligned}$$

Due to symmetry, we obtain

$$\int_{P_1} x^{(i)} x^{(j)} d\sigma_r(x) = - \int_{P_2} x^{(i)} x^{(j)} d\sigma_r(x) = - \int_{P_3} x^{(i)} x^{(j)} d\sigma_r(x) = \int_{P_4} x^{(i)} x^{(j)} d\sigma_r(x).$$

Therefore, we derive

$$\int_{S_r^{d-1}} x^{(i)} x^{(j)} d\sigma_r(x) = \sum_{k=1}^4 \int_{P_k} x^{(i)} x^{(j)} d\sigma_r(x) = 0.$$

To tackle $i = j$, we first observe that

$$1 = \sigma_r(S_r^{d-1}) = \frac{1}{r^2} \int_{S_r^{d-1}} \|x\|^2 d\sigma_r(x) = \frac{1}{r^2} \sum_{i=1}^d \int_{S_r^{d-1}} x^{(i)} x^{(i)} d\sigma_r(x).$$

Due to symmetry, the term $\int_{S_r^{d-1}} x^{(i)} x^{(i)} d\sigma_r(x)$ does not depend on the choice of i and we must therefore have $\int_{S_r^{d-1}} x^{(i)} x^{(i)} d\sigma_r(x) = r^2/d$. According to Corollary 3.11, σ_r is a probabilistic tight frame for \mathbb{R}^d . \square

Remark 3.14. The above proof primarily uses the symmetry of the sphere. Thus, Proposition 3.13 holds for a much larger class of uniform probability measures on symmetric sets K . For instance, it holds for the uniform probability measure on $B_p(r) := \{x \in \mathbb{R}^d : \|x\|_{\ell_p} \leq r\}$ and $\partial B_p(r)$, for $0 < p \leq \infty$.

Next, we construct continuous nonuniform probability measures on the unit circle that form probabilistic unit norm tight frames. Let $\sigma \in \mathcal{M}(\mathcal{B}, S^{d-1})$ represent the uniform probability measure on the circle:

Proposition 3.15. *If $\{x_i\}_{i=1}^n \subset S^1$ is a FUNTF and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function, such that, for all $i = 1, \dots, n$, $y \mapsto f(\langle x_i, y \rangle)$ is measurable and $\int_{S^1} f(\langle x_i, y \rangle) d\sigma(y) = 1$, then the probability measure*

$$(10) \quad \mu(x) = \frac{1}{n} \sum_{i=1}^n f(\langle x_i, x \rangle) \sigma(x)$$

is a probabilistic unit norm tight frame for \mathbb{R}^2 .

Proof. Let $\{x_i\}_{i=1}^n = \left\{ \begin{pmatrix} \cos(\alpha_i) \\ \sin(\alpha_i) \end{pmatrix} : i = 1, \dots, n \right\}$, for $0 \leq \alpha_1, \dots, \alpha_n < 2\pi$. Since, for any $\beta \in [0, 2\pi)$, the collection $\left\{ \begin{pmatrix} \cos(\alpha_i + \beta) \\ \sin(\alpha_i + \beta) \end{pmatrix} : i = 1, \dots, n \right\}$ is a rotation of $\{x_i\}_{i=1}^n$, it also forms a FUNTF, for all $0 \leq \beta \leq 2\pi$. If we parametrize the circle by $[0, 2\pi)$, then the mixture in (10) can be carried over to $[0, 2\pi)$ and may be written as

$$\frac{1}{n} \sum_{i=1}^n f(\cos(\beta - \alpha_i)) d\beta, \quad \beta \in [0, 2\pi),$$

where we have used $x_i = \begin{pmatrix} \cos(\alpha_i) \\ \sin(\alpha_i) \end{pmatrix}$ and $\cos(\beta)\cos(\alpha_i) + \sin(\beta)\sin(\alpha_i) = \cos(\beta - \alpha_i)$. This yields, for any $x \in \mathbb{R}^2$,

$$\begin{aligned} \int_{S^1} |\langle y, x \rangle|^2 d\mu(x) &= \int_0^{2\pi} \left| \left\langle x, \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix} \right\rangle \right|^2 \frac{1}{n} \sum_{i=1}^n f(\cos(\beta - \alpha_i)) d\beta \\ &= \frac{1}{n} \sum_{i=1}^n \int_0^{2\pi} \left| \left\langle x, \begin{pmatrix} \cos(\alpha_i + \beta) \\ \sin(\alpha_i + \beta) \end{pmatrix} \right\rangle \right|^2 f(\cos(\alpha_i + \beta - \alpha_i)) d\beta \\ &= \frac{1}{n} \int_0^{2\pi} f(\cos(\beta)) \sum_{i=1}^n \left| \left\langle x, \begin{pmatrix} \cos(\alpha_i + \beta) \\ \sin(\alpha_i + \beta) \end{pmatrix} \right\rangle \right|^2 d\beta \\ &= \frac{1}{n} \int_0^{2\pi} f(\cos(\beta)) \frac{n}{2} d\beta = \frac{1}{2}. \end{aligned} \quad \square$$

Next, we give an example of Proposition 3.15 that is used in [13] to model the patterns found in granular rod experiments:

Example 3.16. Let $x_0 \in S^1$ and $\kappa > 0$. For the density $f_1(t) = \frac{1}{c_1} \exp(\kappa t)$, we call

$$\mu_1(x) = f_1(\langle x_0, x \rangle) \sigma(x)$$

the *von Mises measure*, which reflects the normal distribution on the circle, see [22]. The constant c normalizes μ such that $\mu(S^1) = 1$. The *Watson measure* $\mu \in \mathcal{M}(\mathcal{B}, S^1)$ is given by

$$\mu_2(x) = f_2(\langle x_0, x \rangle^2) \sigma(x),$$

where $f_2(t) = \frac{1}{c_2} \exp(\kappa t^2)$, and c_2 is a normalizing constant, cf. [22]. For $\kappa > 0$, the density of the Watson measure tends to concentrate around $\pm x_0$, whereas for $\kappa < 0$, the density concentrates around the great circle orthogonal to x_0 . And as $|\kappa|$ increases, the density peaks tighten.

Watson and von Mises measures are widely used in directional statistics. Both densities f_1 and f_2 satisfy the assumptions of Proposition 3.15. Therefore, FUNTF mixtures of von Mises and Watson measures according to (10) form probabilistic unit norm tight frames for \mathbb{R}^2 .

The proof of Proposition 3.15 implicitly relies on the commutativity of the rotation group in \mathbb{R}^2 . The special group in \mathbb{R}^d , for $d > 2$, is not abelian, and we need slightly stronger assumptions. Let G be a finite subgroup of the orthogonal matrices $O(\mathbb{R}^d)$. The G -orbit of $x \in \mathbb{R}^d$ is the collection $\{gx : g \in G\}$. The finite subgroup G is called *irreducible* if the G -orbit of any nonzero $x \in \mathbb{R}^d$ spans \mathbb{R}^d . If $G \subset O(\mathbb{R}^d)$ is an irreducible finite group, then the G -orbit of any nonzero $x \in \mathbb{R}^d$ is a finite tight frame for \mathbb{R}^d , cf. [28]. The latter can be used to verify that the n -th roots of unity, vertices of the platonic solids, and vertices of the truncated icosahedron are finite tight frames, cf. [28]. This construction can also be applied to probability distributions:

Proposition 3.17. *Let G be a finite irreducible subgroup of $O(\mathbb{R}^d)$ and $x_0 \in S^{d-1}$. If $\mu \in \mathcal{M}(\mathcal{B}, S^{d-1})$, then the probability measure*

$$\tilde{\mu}(x) = \frac{1}{|G|} \sum_{g \in G} \mu(g^* x)$$

is a probabilistic unit norm tight frame for \mathbb{R}^d .

Proof. Since $\{gx : g \in G\}$ is a finite tight frame, we obtain

$$\begin{aligned} \int_{S^{d-1}} |\langle y, x \rangle|^2 d\tilde{\mu}(x) &= \frac{1}{|G|} \sum_{g \in G} \int_{S^{d-1}} |\langle y, x \rangle|^2 d\mu(g^*x) \\ &= \frac{1}{|G|} \sum_{g \in G} \int_{S^{d-1}} |\langle y, gx \rangle|^2 d\mu(x) \\ &= \int_{S^{d-1}} \frac{1}{d} \|y\|^2 d\mu(x) = \frac{1}{d} \|y\|^2. \end{aligned} \quad \square$$

3.2. Random Tight Frames for \mathbb{R}^d . The following theorem is the main result of the present paper. Compared to Theorem 2.8, we can replace the uniform distribution with any probabilistic tight frame and the points do not have to be identically distributed. To properly formulate the result, let us recall some notation that we already used in Theorem 2.8. We define $E(Z) := \int_K Z(x) d\mu(x)$, where $Z : K \rightarrow \mathbb{R}^{p \times q}$ is a random matrix/vector that is distributed according to $\mu \in \mathcal{M}(\mathcal{B}, K)$. For notational convenience, we write $Z \in K$ if Z maps into K :

Theorem 3.18. *Let $\{X_k\}_{k=1}^n \subset K$ be a collection of random vectors, independently distributed according to probabilistic tight frames $\{\mu_k\}_{k=1}^n \subset \mathcal{M}(\mathcal{B}, K)$, respectively, whose 4-th moments are finite, i.e., $N_k := \int_K \|y\|^4 d\mu_k(y) < \infty$. If F denotes the random matrix associated to the analysis operator of $\{X_k\}_{k=1}^n$, then we have*

$$(11) \quad E\left(\left\|\frac{1}{n} F^* F - \frac{L}{d} \mathcal{I}_d\right\|_{\mathcal{F}}^2\right) = \frac{1}{n} \left(N - \frac{\tilde{L}}{d}\right),$$

where $L := \frac{1}{n} \sum_{k=1}^n L_k$, $\tilde{L} := \frac{1}{n} \sum_{k=1}^n L_k^2$, $L_k := \int_K \|y\|^2 d\mu_k(y)$, and $N = \frac{1}{n} \sum_{k=1}^n N_k$.

Note that Tyler used FUNTFs to derive M -estimators of multivariate scatter in [20, 26, 27]. Those results are related to the estimation of the population covariance matrix from the sample covariance, and the latter is closely related to Theorem 3.18.

Proof. We observe that the (i, j) -th entry of the random matrix operator $F^* F$ is given by

$$(F^* F)_{i,j} = \sum_{k=1}^n X_k^{(i)} X_k^{(j)},$$

where $X_k = (X_k^{(1)}, \dots, X_k^{(d)})^\top$. First, we fix (i, j) and derive

$$\begin{aligned} E\left(\left(\frac{1}{n} F^* F\right)_{i,j} - \frac{L}{d} \delta_{i,j}\right)^2 &= E\left(\frac{1}{n^2} \sum_{k,l} X_k^{(i)} X_k^{(j)} X_l^{(i)} X_l^{(j)} - \frac{2L}{d} \delta_{i,j} \frac{1}{n} \sum_{k=1}^n X_k^{(i)} X_k^{(j)} + \frac{L^2}{d^2} \delta_{i,j}\right) \\ &= \frac{1}{n^2} \sum_{k=1}^n E(X_k^{(i)} X_k^{(j)} X_k^{(i)} X_k^{(j)}) + \frac{1}{n^2} \sum_{k \neq l} E(X_k^{(i)} X_k^{(j)} X_l^{(i)} X_l^{(j)}) \\ &\quad - \frac{2L}{d} \delta_{i,j} \frac{1}{n} \sum_{k=1}^n E(X_k^{(i)} X_k^{(j)}) + \frac{L^2}{d^2} \delta_{i,j}. \end{aligned}$$

Let us denote $M_k(i, j) := \int_K |y^{(i)}|^2 |y^{(j)}|^2 d\mu_k(y)$ and $M = \frac{1}{n} \sum_{k=1}^n M_k$. Since the random vectors are independent and the measures $\{\mu_k\}_{k=1}^n$ satisfy (9), we obtain

$$\begin{aligned}
E\left(\left(\frac{1}{n} F^* F\right)_{i,j} - \frac{L}{d} \delta_{i,j}\right)^2 &= \frac{1}{n^2} \sum_{k=1}^n M_k(i, j) + \frac{1}{n^2} \sum_{k \neq l} \frac{L_k}{d} \frac{L_l}{d} \delta_{i,j} - \frac{2L}{d} \frac{L}{d} + \frac{L^2}{d^2} \delta_{i,j} \\
&= \frac{1}{n^2} \sum_{k=1}^n M_k(i, j) + \frac{1}{nd^2} \sum_{k=1}^n L_k (L - \frac{1}{n} L_k) \delta_{i,j} - \frac{L^2}{d^2} \delta_{i,j} \\
&= \frac{1}{n^2} \sum_{k=1}^n M_k(i, j) + \frac{L^2}{d^2} \delta_{i,j} - \frac{\tilde{L}}{nd^2} \delta_{i,j} - \frac{L^2}{d^2} \delta_{i,j} \\
&= \frac{1}{n^2} \sum_{k=1}^n M_k(i, j) - \frac{\tilde{L}}{nd^2} \delta_{i,j} \\
(12) \quad &= \frac{1}{n} (M_{i,j} - \frac{\tilde{L}}{d^2} \delta_{i,j}).
\end{aligned}$$

The Frobenius norm $\|A\|_{\mathcal{F}}$ of a matrix $A = (a_{i,j})_{i,j}$ equals $(\sum_{i,j} a_{i,j}^2)^{1/2}$. Since $\frac{1}{n} F^* F - \frac{L}{d} \mathcal{I}_d$ is a $d \times d$ matrix, we obtain

$$E\left(\left\|\frac{1}{n} F^* F - \frac{L}{d} \mathcal{I}_d\right\|_{\mathcal{F}}^2\right) = \frac{1}{n} \left(\frac{1}{n} \sum_{k=1}^n \int_K \|y\|^4 d\mu_k(y) - \frac{\tilde{L}}{d}\right). \quad \square$$

If the N_k in Theorem 3.18 are bounded by a universal constant, then (11) essentially decays as $\frac{1}{n}$. The smaller N the faster tends (11) to zero. In other words, the 4-th moments specify the exact decay.

Let us present few examples that lead to asymptotic tight frames:

Example 3.19. We have already pointed out in Remark 3.14 that uniform probability measures on ℓ_p -balls and ℓ_p -spheres, for $0 < p \leq \infty$, form probabilistic tight frames. According to Theorem 3.18, i.i.d. random points according to the latter distributions approximate a tight frame.

Remark 3.20. Vershynin has derived a result about the approximation of covariance matrices that is similar to Theorem 3.18. His statement is about convergence with high probability in the operator norm. The approximation error is then estimated by a constant times $(\frac{1}{n})^{1/2-2/q}$, where all $\{\mu_k\}_{k=1}^n$ must have finite q -th moments and $q > 4$, cf. Theorem 6.1 in [29]. For sub-Gaussian distributions, i.e., for μ such that, for some $s > 0$,

$$\mu(|\langle X, x \rangle| > t) \leq 2e^{-\frac{t^2}{s^2}}, \quad \text{for } t > 0 \text{ and } x \in S^{d-1},$$

where X is distributed according to μ , Vershynin can estimate the approximation error in the operator norm by a constant times $(\frac{1}{n})^{1/2}$, cf. Proposition 2.1 in [29]. Note that the latter matches our decay rates for the mean squared error (we squared the Frobenius norm). Nevertheless, our results address more general distributions since Theorem 3.18 only requires that the 4-th moments exist. We do not have any assumption on higher moments, and we do not require that the distributions are sub-Gaussian.

For probabilistic unit norm tight frames, Theorem 3.18 simplifies as follows:

Corollary 3.21. *Let $\{X_k\}_{k=1}^n \subset S^{d-1}$ be a collection of random vectors, independently distributed according to probabilistic unit norm tight frames $\{\mu_k\}_{k=1}^n \subset \mathcal{M}(\mathcal{B}, S^{d-1})$, respectively. If F denotes the random matrix associated to the analysis operator of $\{X_k\}_{k=1}^n$, then*

$$(13) \quad E(\|\frac{1}{n}F^*F - \frac{1}{d}\mathcal{I}_d\|_{\mathcal{F}}^2) = \frac{1}{n}(1 - \frac{1}{d}).$$

Randomness is used in compressed sensing to design suitable measurements matrices. Each row of such random matrices is a random vector whose covariance must usually be close to the identity matrix. The construction of random vectors in compressed sensing is commonly based on Bernoulli, Gaussian, and sub-Gaussian distributions. We shall explain that these random vectors are induced by probabilistic tight frames, and in fact, we can apply Theorem 3.18:

Example 3.22. Let $\{X_k\}_{k=1}^n$ be a collection of d -dimensional random vectors such that each vector's entries are i.i.d according to a probability measure with zero mean and finite 4-th moments. This implies that each X_k is distributed with respect to a probabilistic tight frame whose 4-th moments exist. Thus, the assumptions in Theorem 3.18 are satisfied, and we can compute (11) for some specific distributions that are related to compressed sensing:

- If the entries of X_k , $k = 1, \dots, n$, are i.i.d. according to a Bernoulli distribution that takes the values $\pm \frac{1}{\sqrt{d}}$ with probability $\frac{1}{2}$, then X_k is distributed according to a normalized counting measure supported on the vertices of the d -dimensional hypercube. Thus, X_k is distributed according to a probabilistic unit norm tight frame for \mathbb{R}^d , cf. Remark 3.14, and Corollary 3.21 can be applied.
- If the entries of X_k , $k = 1, \dots, n$, are i.i.d. according to a Gaussian distribution with 0 mean and variance $\frac{1}{d}$, then X_k is distributed according to a multivariate Gaussian probability measure $\mu \in \mathcal{M}(\mathcal{B}, \mathbb{R}^d)$ whose covariance matrix is $\frac{1}{d}\mathcal{I}_d$, and μ forms a probabilistic tight frame for \mathbb{R}^d . Since the moments of a multivariate Gaussian random vector are well-known, we can explicitly compute $N = 1 + \frac{2}{d}$, $L = 1$, and $\tilde{L} = 1$ in Theorem 3.18. Thus, the right-hand side of (11) equals $\frac{1}{n}(1 + \frac{1}{d})$.
- If the entries of X_k , $k = 1, \dots, n$, are i.i.d. with respect to a sub-Gaussian probability measure with 0 mean, then X_k is distributed according to a probabilistic tight frame for \mathbb{R}^d that has finite moments, and Theorem 3.18 can be applied.

Remark 3.23. When compressed sensing is applied to MRI, the rows of the discrete Fourier matrix $W = (\frac{\omega^{jk}}{\sqrt{d}})_{j,k=0}^{d-1}$, where $\omega = e^{\frac{-2\pi i}{d}}$ and $i^2 = -1$, are usually subsampled to reduce acquisition time. A uniform subsampling of the discrete Fourier matrix is induced by a (complex) probabilistic tight frame: The entire machinery of probabilistic frames for \mathbb{R}^d developed in Section 3.1 can be extended to probabilistic frames for \mathbb{C}^d in a straight-forward manner. Synthesis, analysis, and frame operator can be analogously defined, and a probability measure μ on $K \subset \mathbb{C}^d$ is then a probabilistic tight frame for \mathbb{C}^d if and only if its “second moments” satisfy

$$\int_K z^{(i)} \overline{z^{(j)}} d\mu(z) = \frac{1}{d} \delta_{i,j} \int_K \|z\|^2 d\mu(z).$$

Let $\{Z_k\}_{k=1}^n$ be a collection of random vectors that are i.i.d. according to a normalized counting measure μ supported on the row vectors of the discrete Fourier matrix. Since W is unitary and the absolute value of each entry is $\frac{1}{\sqrt{d}}$, the latter measure is a probabilistic tight frame for \mathbb{C}^d , and its “4-th moments” satisfy $\int_K |z^{(i)}|^2 |z^{(j)}|^2 d\mu(z) = \frac{1}{d^2}$. Corollary 3.21 can also be extended to probabilistic tight frames for \mathbb{C}^d .

We conclude this section by rephrasing Theorem 3.18 in terms of general probability distributions on $K \subset \mathbb{R}^d$ that are not necessarily tight frames. For a matrix $U = (u_{i,j}) \in \mathbb{R}^{d \times d}$, we denote $\|U\|_1 := \sum_{i,j} |u_{i,j}|$:

Theorem 3.24. *Let $\{X_k\}_{k=1}^n \subset K$ be a collection of random vectors that are independently distributed according to probability measures $\{\mu_k\}_{k=1}^n \subset \mathcal{M}(\mathcal{B}, K)$, respectively, whose 4-th moments are finite, i.e., $N_k := \int_K \|y\|^4 d\mu_k(y) < \infty$. Let $\{S_k\}_{k=1}^n$ be the frame operators of $\{\mu_k\}_{k=1}^n$, respectively. If F denotes the random matrix associated to the analysis operator of $\{X_k\}_{k=1}^n$, then we have*

$$E(\|\frac{1}{n}F^*F - S\|_{\mathcal{F}}^2) = \frac{1}{n}(N - \frac{\|\tilde{S}\|_1}{d^2}),$$

where $S = \frac{1}{n} \sum_{k=1}^n S_k$, $\tilde{S}_{i,j} = \frac{1}{n} \sum_{k=1}^n ((S_k)_{i,j})^2$, and $N = \frac{1}{n} \sum_{k=1}^n N_k$.

For instance, Theorem 3.24 applies to random vectors that have a multivariate sub-Gaussian distribution and whose entries are not necessarily independent. The proof can be derived by following the lines of the proof of Theorem 3.18 while replacing $\frac{L_k}{d}$ with S_k .

4. THE PROBABILISTIC FRAME POTENTIAL

4.1. Minimizing the Probabilistic Frame Potential. The minimizers of the frame potential are the configurations of n points on the sphere that form a FUNTF. What happens if we have to distribute a continuous mass on the sphere S^{d-1} or, more general, on $K \subset \mathbb{R}^d$?

Definition 4.1. For $0 \notin K$ and $\mu \in \mathcal{M}(\mathcal{B}, K)$, we call

$$(14) \quad \text{PFP}(\mu) = \frac{\int_K \int_K |\langle x, y \rangle|^2 d\mu(x) d\mu(y)}{(\int_K \|x\|^2 d\mu(x))^2}$$

the *probabilistic frame potential* of μ .

We easily observe that $\text{supp}(\mu) \neq \{0\}$ if and only if $\int_K \|x\|^2 d\mu(x) \neq 0$. Therefore, $\text{PFP}(\mu)$ in (14) is well-defined. We aim to characterize the minimizers of the probabilistic frame potential for fixed K . In fact, these minimizers are the probabilistic tight frames provided that the latter exist for the particular choice of K . The following theorem generalizes Theorem 2.7:

Theorem 4.2. *If $0 \notin K$ and $\mu \in \mathcal{M}(\mathcal{B}, K)$, then*

$$(15) \quad \text{PFP}(\mu) \geq \frac{1}{d},$$

and equality holds if and only if μ is a probabilistic tight frame for \mathbb{R}^d .

Proof. Let $m_{i,j}(\mu)$ denote the second moments of μ , i.e., $m_{i,j}(\mu) = \int_K x^{(i)} x^{(j)} d\mu(x)$. We obtain

$$(16) \quad \int_K \|x\|^2 d\mu(x) = \sum_{i=1}^d \int_K x^{(i)} x^{(i)} d\mu(x) = \sum_{i=1}^d m_{i,i}(\mu).$$

The probabilistic frame potential can be written as

$$\begin{aligned} \text{PFP}(\mu) &= \frac{\int_K \int_K \sum_{i=1}^d \sum_{j=1}^d x^{(i)} y^{(i)} x^{(j)} y^{(j)} d\mu(x) d\mu(y)}{\sum_{i=1}^d m_{i,i}(\mu)} \\ &= \frac{\sum_{i=1}^d \sum_{j=1}^d m_{i,j}^2(\mu)}{\sum_{i=1}^d m_{i,i}(\mu)}. \end{aligned}$$

The Hölder inequality implies

$$(17) \quad \sum_{i=1}^d m_{i,i}(\mu) \leq \left(\sum_{i=1}^d m_{i,i}^2(\mu) \right)^{1/2} \left(\sum_{i=1}^d 1 \right)^{1/2} \leq \left(\sum_{i=1}^d \sum_{j=1}^d m_{i,j}^2(\mu) \right)^{1/2} d^{1/2},$$

which yields (15).

Next, assume that the latter inequalities (17), in fact, are equalities. This requires $m_{i,j}(\mu) = 0$, for all $i \neq j$, and the Hölder inequality was actually an equality. The Hölder inequality becomes an equality if and only if the occurring sequences are linearly dependent. Thus, $(m_{i,i}(\mu))_{i=1}^d$ must be a multiple of the constant sequence. Due to (16), we obtain $m_{i,i}(\mu) = \frac{1}{d} \int_K \|x\|^2 d\mu(x)$, for all $i = 1, \dots, d$, and hence μ is a probabilistic tight frame, cf. Corollary 3.11.

Conversely, if μ is a probabilistic tight frame, then $m_{i,j}(\mu) = \delta_{i,j} \frac{1}{d} \int_K \|x\|^2 d\mu(x)$ due to Corollary 3.11. Thus, we have equality in (17) and hence in (15). \square

According to Proposition 3.13, probabilistic tight frames exist for $K = S^{d-1}$. If $K = \mathbb{R}^d \setminus \{0\}$, then the normalized counting measure of any finite tight frame is a probabilistic tight frame. Hence, Theorem 4.2 leads to the following generalization of Theorem 2.6 and Theorem 2.7:

Corollary 4.3. *If $K = S^{d-1}$, then the minimizers of the probabilistic frame potential are exactly the probabilistic unit norm tight frames for \mathbb{R}^d . If $K = \mathbb{R}^d \setminus \{0\}$, then the minimizers of the probabilistic frame potential are exactly the probabilistic tight frames for \mathbb{R}^d .*

Let us explore the relations between Corollary 4.3 and the discrete frame potential in Theorem 2.6. For fixed d and $K = S^{d-1}$, every FUNTF induces a minimizer of the probabilistic frame potential:

Example 4.4. If $\{x_i\}_{i=1}^n \subset S^{d-1}$ is a FUNTF, then $\text{FP}(\{x_i\}_{i=1}^n) = \frac{n^2}{d}$ according to Theorem 2.6. Thus, the discrete point measure $\frac{1}{n} \mu_{x_1, \dots, x_n}$ satisfies $\text{PFP}(\frac{1}{n} \mu_{x_1, \dots, x_n}) = \frac{1}{d}$, and therefore is a minimizer of the probabilistic frame potential for $K = S^{d-1}$.

Contrary to Theorem 2.6, orthonormal systems that are not a basis, do not induce a minimizer:

Example 4.5. Let $\{x_i\}_{i=1}^n \subset S^{d-1}$ be an orthonormal system with $n < d$. Due to Theorem 2.6, we have $\text{FP}(\{x_i\}_{i=1}^n) = n$. For $K = S^{d-1}$, this implies $\text{PFP}(\frac{1}{n} \mu_{x_1, \dots, x_n}) = \frac{n}{n^2}$. Since $n < d$, we deduce $\text{PFP}(\frac{1}{n} \mu_{x_1, \dots, x_n}) = \frac{1}{n} > \frac{1}{d}$.

4.2. Relations to Spherical t -designs. Let σ denote the uniform probability measure on S^{d-1} . A *spherical t -design* is a finite subset $\{x_i\}_{i=1}^n \subset S^{d-1}$, such that,

$$\frac{1}{n} \sum_{i=1}^n h(x_i) = \int_{S^{d-1}} h(x) d\sigma(x),$$

for all homogeneous polynomials h of total degree less than or equal to t in d variables, cf. [9]. We call a probability measure $\mu \in \mathcal{M}(\mathcal{B}, S^{d-1})$ a *probabilistic spherical t -design* if

$$(18) \quad \int_{S^{d-1}} h(x) d\mu(x) = \int_{S^{d-1}} h(x) d\sigma(x),$$

for all homogeneous polynomials h with total degree less than or equal to t .

Theorem 4.6. *If $\mu \in \mathcal{M}(\mathcal{B}, S^{d-1})$, then the following are equivalent:*

- (i) μ is a probabilistic spherical 2-design.
- (ii) μ minimizes

$$(19) \quad \frac{\int_{S^{d-1}} \int_{S^{d-1}} |\langle x, y \rangle|^2 d\mu(x) d\mu(y)}{\int_{S^{d-1}} \int_{S^{d-1}} \|x - y\|^2 d\mu(x) d\mu(y)}$$

among all probability measures $\mathcal{M}(\mathcal{B}, S^{d-1})$.

- (iii) μ satisfies

$$(20) \quad \int_{S^{d-1}} x d\mu(x) = 0$$

$$(21) \quad \int_{S^{d-1}} x^{(i)} x^{(j)} d\mu(x) = \frac{1}{d} \delta_{i,j}.$$

In particular, if μ is a probabilistic unit norm tight frame, then $\nu(A) := \frac{1}{2}(\mu(A) + \mu(-A))$, for $A \in \mathcal{B}$, defines a probabilistic spherical 2-design.

Proof. To show that (i) and (iii) are equivalent, we observe that the uniform probability measure σ is a probabilistic unit norm tight frame, cf. Proposition 3.13. It hence satisfies (21) according to Corollary 3.11. Due to its symmetry, σ also satisfies (20). Thus according to (18), the probabilistic spherical 2-designs are exactly those probability measures $\mu \in \mathcal{M}(\mathcal{B}, S^{d-1})$ that satisfy (20) and (21).

To address the equivalence between (ii) and (iii), we will observe that the minimization (19) splits into minimizing its numerator and maximizing its denominator. Due to Corollary 3.11 and Corollary 4.3, the numerator is minimized if and only if μ satisfies (21). Let us rewrite the denominator as follows:

$$\begin{aligned} \int_{S^{d-1}} \int_{S^{d-1}} \|x - y\|^2 d\mu(x) d\mu(y) &= \int_{S^{d-1}} \int_{S^{d-1}} \sum_{i=1}^d x^{(i)} x^{(i)} + y^{(i)} y^{(i)} - 2x^{(i)} y^{(i)} d\mu(x) d\mu(y) \\ &= \int_{S^{d-1}} \int_{S^{d-1}} 2d\mu(x) d\mu(y) - 2 \sum_{i=1}^d \int_{S^{d-1}} \int_{S^{d-1}} x^{(i)} y^{(i)} d\mu(x) d\mu(y) \\ &= 2 - 2 \sum_{i=1}^d \left(\int_{S^{d-1}} x^{(i)} d\mu(x) \right)^2. \end{aligned}$$

It is hence maximized if and only if $\int_{S^{d-1}} x d\mu(x) = 0$. Thus, (iii) implies (ii). For the reverse implication, we need to verify that there is a probability measure that minimizes the numerator and maximizes the denominator of (19) at the same time.

We first recall that probabilistic unit norm tight frames exist, cf. Proposition 3.13. If μ is such a probabilistic unit norm tight frame, then ν as defined in Theorem 4.6 satisfies (20), and ν also satisfies (21) since its second moments coincide with those of μ . Hence, (ii) implies (iii), and we can conclude the proof. \square

Remark 4.7. We have shown in the proof of Theorem 4.6 that the maximizers of $\int_{S^{d-1}} \int_{S^{d-1}} \|x - y\|^2 d\mu(x) d\mu(y)$ are exactly the zero mean probability measures on the sphere. The latter result is already implicitly contained in a work by Bjoerck [3], in which he considers the integrals over the unit ball and then shows that the mass of the maximizer must completely be contained in the unit sphere.

5. CONCLUSIONS

First, we introduced probabilistic frames and verified that many properties from finite frames can be adopted. Secondly, we used probabilistic tight frames to significantly improve a result by Goyal, Vetterli, and Thao in [18] about the random choice of points on the sphere. We still approximate a tight frame while allowing for a much wider class of probability measures, namely any probabilistic tight frame. The requirement of identical distributions is also removed. We also verified that many random matrices, which are used in compressed sensing, are induced by probabilistic tight frames. Thirdly, we extended results about the frame potential as introduced by Benedetto and Fickus in [2]. In fact, we demonstrated that probabilistic tight frames are the minimizers of the probabilistic frame potential.

ACKNOWLEDGEMENTS

The author was supported by the Intramural Research Program of the National Institute of Child Health and Human Development and by NIH/DFG Research Career Transition Awards Program (EH 405/1-1/575910).

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